

ON A RECONSTRUCTION THEOREM FOR HOLONOMIC SYSTEMS

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ABSTRACT. Let X be a complex manifold. The classical Riemann-Hilbert correspondence associates to a regular holonomic system \mathcal{M} the \mathbb{C} -constructible complex of its holomorphic solutions. Let t be the affine coordinate in the complex projective line. If \mathcal{M} is not necessarily regular, we associate to it the ind- \mathbb{R} -constructible complex G of tempered holomorphic solutions to $\mathcal{M} \boxtimes \mathcal{D}e^t$. We conjecture that this provides a Riemann-Hilbert correspondence for holonomic systems. We discuss the functoriality of this correspondence, we prove that \mathcal{M} can be reconstructed from G if $\dim X = 1$, and we show how the Stokes data are encoded in G .

INTRODUCTION

Let X be a complex manifold. The Riemann-Hilbert correspondence of [3] establishes an anti-equivalence

$$\mathbf{D}_{\text{r-hol}}^b(\mathcal{D}_X) \xrightleftharpoons[\Psi^0]{\Phi^0} \mathbf{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X)$$

between regular holonomic \mathcal{D} -modules and \mathbb{C} -constructible complexes. Here, $\Phi^0(\mathcal{L}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, \mathcal{O}_X)$ is the complex of holomorphic solutions to \mathcal{L} , and $\Psi^0(L) = \mathcal{T}\mathcal{H}om(L, \mathcal{O}_X) = R\mathcal{H}om(L, \mathcal{O}_X^\dagger)$ is the complex of holomorphic functions tempered along L . Since $\mathcal{L} \simeq \Psi^0(\Phi^0(\mathcal{L}))$, this shows in particular that \mathcal{L} can be reconstructed from $\Phi^0(\mathcal{L})$.

We are interested here in holonomic \mathcal{D} -modules which are not necessarily regular.

The theory of ind-sheaves from [7] allows one to consider the complex $\Phi^t(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X^\dagger)$ of tempered holomorphic solutions to a holonomic module \mathcal{M} . The basic example $\Phi^t(\mathcal{D}_{\mathbb{C}}e^{1/x})$ was computed in [8], and the functor Φ^t has been studied in [11, 12]. However, since $\Phi^t(\mathcal{D}_{\mathbb{C}}e^{1/x}) \simeq \Phi^t(\mathcal{D}_{\mathbb{C}}e^{2/x})$, one cannot reconstruct \mathcal{M} from $\Phi^t(\mathcal{M})$.

Set $\Phi(\mathcal{M}) = \Phi^t(\mathcal{M} \boxtimes \mathcal{D}_{\mathbb{P}}e^t)$, for t the affine variable in the complex projective line \mathbb{P} . This is an ind- \mathbb{R} -constructible complex in $X \times \mathbb{P}$.

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The arguments in [1] suggested us how \mathcal{M} could be reconstructed from $\Phi(\mathcal{M})$ via a functor Ψ , described below (§3).

We conjecture that the contravariant functors

$$\mathbf{D}^b(\mathcal{D}_X) \xrightleftharpoons[\Psi]{\Phi} \mathbf{D}^b(\mathbb{IC}_{X \times \mathbb{P}}),$$

between the derived categories of \mathcal{D}_X -modules and of ind-sheaves on $X \times \mathbb{P}$, provide a Riemann-Hilbert correspondence for holonomic systems.

To corroborate this statement, we discuss the functoriality of Φ and Ψ with respect to proper direct images and to tensor products with regular objects (§4). This allows a reduction to holonomic modules with a good formal structure.

When X is a curve and \mathcal{M} is holonomic, we prove that the natural morphism $\mathcal{M} \rightarrow \Psi(\Phi(\mathcal{M}))$ is an isomorphism (§6). Thus \mathcal{M} can be reconstructed from $\Phi(\mathcal{M})$.

Recall that irregular holonomic modules are subjected to the Stokes phenomenon. We describe with an example how the Stokes data of \mathcal{M} are encoded topologically in the ind- \mathbb{R} -constructible sheaf $\Phi(\mathcal{M})$ (§7).

In this Note, the proofs are only sketched. Details will appear in a forthcoming paper. There, we will also describe some of the properties of the essential image of holonomic systems by the functor Φ . Such a category is related to a construction of [14].

1. NOTATIONS

We refer to [5, 7, 4].

Let X be a real analytic manifold.

Denote by $\mathbf{D}^b(\mathbb{C}_X)$ the bounded derived category of sheaves of \mathbb{C} -vector spaces, and by $\mathbf{D}_{\mathbb{R}\text{-c}}^b(\mathbb{C}_X)$ the full subcategory of objects with \mathbb{R} -constructible cohomologies. Denote by $\otimes, R\mathcal{H}om, f^{-1}, Rf_*, Rf_!, f^!$ the six Grothendieck operations for sheaves. (Here $f: X \rightarrow Y$ is a morphism of real analytic manifolds.)

For $S \subset X$ a locally closed subset, we denote by \mathbb{C}_S the zero extension to X of the constant sheaf on S .

Recall that an ind-sheaf is an ind-object in the category of sheaves with compact support. Denote by $\mathbf{D}^b(\mathbb{IC}_X)$ the bounded derived category of ind-sheaves, and by $\mathbf{D}_{\mathbb{R}\text{-c}}^b(\mathbb{IC}_X)$ the full subcategory of objects with ind- \mathbb{R} -constructible cohomologies. Denote by $\otimes, R\mathcal{I}\mathcal{H}om, f^{-1}, Rf_*, Rf_{!!}, f^!$ the six Grothendieck operations for ind-sheaves.

Denote by α the left adjoint of the embedding of sheaves into ind-sheaves. One has $\alpha(\varinjlim F_i) = \varinjlim F_i$. Denote by β the left adjoint of α .

Denote by $\mathcal{D}b_X^t$ the ind- \mathbb{R} -constructible sheaf of tempered distributions.

Let X be a complex manifold. We set for short $d_X = \dim X$.

Denote by \mathcal{O}_X and \mathcal{D}_X the rings of holomorphic functions and of differential operators. Denote by Ω_X the invertible sheaf of differential forms of top degree.

Denote by $\mathbf{D}^b(\mathcal{D}_X)$ the bounded derived category of left \mathcal{D}_X -modules, and by $\mathbf{D}_{\text{hol}}^b(\mathcal{D}_X)$ and $\mathbf{D}_{\text{r-hol}}^b(\mathcal{D}_X)$ the full subcategories of objects with holonomic and regular holonomic cohomologies, respectively. Denote by \otimes^D , Df^{-1} , Df_* the operations for \mathcal{D} -modules. (Here $f: X \rightarrow Y$ is a morphism of complex manifolds.)

Denote by $\mathbb{D}\mathcal{M}$ the dual of \mathcal{M} (with shift such that $\mathbb{D}\mathcal{O}_X \simeq \mathcal{O}_X$).

For $Z \subset X$ a closed analytic subset, we denote by $R\Gamma_{[Z]}\mathcal{M}$ and $\mathcal{M}(*Z)$ the relative algebraic cohomologies of a \mathcal{D}_X -module \mathcal{M} .

Denote by $\text{ss}(\mathcal{M}) \subset X$ the singular support of \mathcal{M} , that is the set of points where the characteristic variety is not reduced to the zero-section.

Denote by $\mathcal{O}_X^t \in \mathbf{D}_{\text{IR-c}}^b(\text{IC}_X)$ the complex of tempered holomorphic functions. Recall that \mathcal{O}_X^t is the Dolbeault complex of $\mathcal{D}b_X^t$ and that it has a structure of $\beta\mathcal{D}_X$ -module. We will write for short $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X^t)$ instead of $R\mathcal{I}\mathcal{H}om_{\beta\mathcal{D}_X}(\beta\mathcal{M}, \mathcal{O}_X^t)$.

2. EXPONENTIAL \mathcal{D} -MODULES

Let X be a complex analytic manifold. Let $D \subset X$ be a hypersurface, and set $U = X \setminus D$. For $\varphi \in \mathcal{O}_X(*D)$, we set

$$\begin{aligned}\mathcal{D}_X e^\varphi &= \mathcal{D}_X / \{P : Pe^\varphi = 0 \text{ on } U\}, \\ \mathcal{E}_{D|X}^\varphi &= (\mathcal{D}_X e^\varphi)(*D).\end{aligned}$$

As an $\mathcal{O}_X(*D)$ -module, $\mathcal{E}_{D|X}^\varphi$ is generated by e^φ . Note that $\text{ss}(\mathcal{E}_{D|X}^\varphi) = D$, and $\mathcal{E}_{D|X}^\varphi$ is holonomic. It is regular if $\varphi \in \mathcal{O}_X$, since then $\mathcal{E}_{D|X}^\varphi \simeq \mathcal{O}_X(*D)$.

One easily checks that $(\mathbb{D}\mathcal{E}_{D|X}^\varphi)(*D) \simeq \mathcal{E}_{D|X}^{-\varphi}$.

Proposition 2.1. *If $\dim X = 1$, and φ has an effective pole at every point of D , then $\mathbb{D}\mathcal{E}_{D|X}^\varphi \simeq \mathcal{E}_{D|X}^{-\varphi}$.*

Let \mathbb{P} be the complex projective line and denote by t the coordinate on $\mathbb{C} = \mathbb{P} \setminus \{\infty\}$.

For $c \in \mathbb{R}$, we set for short

$$\begin{aligned}\{\text{Re } \varphi < c\} &= \{x \in U : \text{Re } \varphi(x) < c\}, \\ \{\text{Re}(t + \varphi) < c\} &= \{(x, t) : x \in U, t \in \mathbb{C}, \text{Re}(t + \varphi(x)) < c\}.\end{aligned}$$

Consider the ind- \mathbb{R} -constructible sheaves on X and on $X \times \mathbb{P}$, respectively,

$$\begin{aligned}\mathbb{C}_{\{\text{Re } \varphi < ?\}} &= \varinjlim_{c \rightarrow +\infty} \mathbb{C}_{\{\text{Re } \varphi < c\}}, \\ \mathbb{C}_{\{\text{Re}(t + \varphi) < ?\}} &= \varinjlim_{c \rightarrow +\infty} \mathbb{C}_{\{\text{Re}(t + \varphi) < c\}}.\end{aligned}$$

The following result is analogous to [1, Proposition 7.1]. Its proof is simpler than loc. cit., since φ is differentiable.

Proposition 2.2. *One has an isomorphism in $\mathbf{D}^b(\mathcal{D}_X)$*

$$\mathcal{E}_{D|X}^\varphi \xrightarrow{\sim} Rq_* R\mathcal{H}om_{p^{-1}\mathcal{D}_{\mathbb{P}}}(p^{-1}\mathcal{E}_{\infty|\mathbb{P}}^t, R\mathcal{H}om(\mathbb{C}_{\{\operatorname{Re}(t+\varphi)<?\}}, \mathcal{O}_{X \times \mathbb{P}}^t)),$$

for q and p the projections from $X \times \mathbb{P}$.

The following result is analogous to [8, Proposition 7.3].

Lemma 2.3. *Denote by (u, v) the coordinates in \mathbb{C}^2 . There is an isomorphism in $\mathbf{D}^b(\operatorname{IC}_{\mathbb{C}^2})$*

$$R\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^2}}(\mathcal{E}_{\{v=0\}|\mathbb{C}^2}^{u/v}, \mathcal{O}_{\mathbb{C}^2}^t) \simeq R\mathcal{I}\mathcal{H}om(\mathbb{C}_{\{v \neq 0\}}, \mathbb{C}_{\{\operatorname{Re} u/v < ?\}}).$$

Proposition 2.4. *There is an isomorphism in $\mathbf{D}^b(\operatorname{IC}_X)$*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}\mathcal{E}_{D|X}^{-\varphi}, \mathcal{O}_X^t) \simeq R\mathcal{I}\mathcal{H}om(\mathbb{C}_U, \mathbb{C}_{\{\operatorname{Re} \varphi < ?\}}).$$

Proof. As $\mathbb{D}\mathcal{E}_{\{v=0\}|\mathbb{C}^2}^{u/v} \simeq \mathcal{E}_{\{v=0\}|\mathbb{C}^2}^{-u/v}$, Lemma 2.3 gives

$$\Omega_{\mathbb{C}^2}^t \otimes_{\mathcal{D}_{\mathbb{C}^2}}^L \mathcal{E}_{\{v=0\}|\mathbb{C}^2}^{-u/v}[-2] \simeq R\mathcal{I}\mathcal{H}om(\mathbb{C}_{\{v \neq 0\}}, \mathbb{C}_{\{\operatorname{Re} u/v < ?\}}).$$

Write $\varphi = a/b$ for $a, b \in \mathcal{O}_X$ such that $b^{-1}(0) \subset D$, and consider the map

$$f = (a, b): X \rightarrow \mathbb{C}^2.$$

As $\mathbf{D}f^{-1}\mathcal{E}_{\{v=0\}|\mathbb{C}^2}^{-u/v} \simeq \mathcal{E}_{D|X}^{-\varphi}$, [7, Theorem 7.4.1] implies

$$\Omega_X^t \otimes_{\mathcal{D}_X}^L \mathcal{E}_{D|X}^{-\varphi}[-d_X] \simeq R\mathcal{I}\mathcal{H}om(\mathbb{C}_U, \mathbb{C}_{\{\operatorname{Re} \varphi < ?\}}).$$

Finally, one has

$$\Omega_X^t \otimes_{\mathcal{D}_X}^L \mathcal{E}_{D|X}^{-\varphi}[-d_X] \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}\mathcal{E}_{D|X}^{-\varphi}, \mathcal{O}_X^t).$$

□

3. A CORRESPONDENCE

Let X be a complex analytic manifold. Recall that \mathbb{P} denotes the complex projective line. Consider the contravariant functors

$$\mathbf{D}^b(\mathcal{D}_X) \xrightleftharpoons[\Psi]{\Phi} \mathbf{D}^b(\operatorname{IC}_{X \times \mathbb{P}})$$

defined by

$$\begin{aligned} \Phi(\mathcal{M}) &= R\mathcal{H}om_{\mathcal{D}_{X \times \mathbb{P}}}(\mathcal{M} \boxtimes^{\mathbf{D}} \mathcal{E}_{\infty|\mathbb{P}}^t, \mathcal{O}_{X \times \mathbb{P}}^t), \\ \Psi(F) &= Rq_* R\mathcal{H}om_{p^{-1}\mathcal{D}_{\mathbb{P}}}(p^{-1}\mathcal{E}_{\infty|\mathbb{P}}^t, R\mathcal{H}om(F, \mathcal{O}_{X \times \mathbb{P}}^t)), \end{aligned}$$

for q and p the projections from $X \times \mathbb{P}$.

We conjecture that this provides a Riemann-Hilbert correspondence for holonomic systems:

Conjecture 3.1. (i) *The natural morphism of endofunctors of $\mathbf{D}^b(\mathcal{D}_X)$*

$$(3.1) \quad \text{id} \rightarrow \Psi \circ \Phi$$

is an isomorphism on $\mathbf{D}_{\text{hol}}^b(\mathcal{D}_X)$.

(ii) *The restriction of Φ*

$$\Phi|_{\mathbf{D}_{\text{hol}}^b(\mathcal{D}_X)}: \mathbf{D}_{\text{hol}}^b(\mathcal{D}_X) \rightarrow \mathbf{D}^b(\mathbb{IC}_{X \times \mathbb{P}})$$

is fully faithful.

Let us prove some results in this direction.

4. FUNCTORIAL PROPERTIES

The next two Propositions are easily deduced from the results in [7].

Proposition 4.1. *Let $f: X \rightarrow Y$ be a proper map, and set $f_{\mathbb{P}} = f \times \text{id}_{\mathbb{P}}$. Let $\mathcal{M} \in \mathbf{D}_{\text{hol}}^b(\mathcal{D}_X)$ and $F \in \mathbf{D}_{\text{IR-c}}^b(\mathbb{IC}_{X \times \mathbb{P}})$. Then*

$$\begin{aligned} \Phi(\mathbb{D}f_*\mathcal{M}) &\simeq Rf_{\mathbb{P}!!}\Phi(\mathcal{M})[d_X - d_Y], \\ \Psi(Rf_{\mathbb{P}!!}F) &\simeq \mathbb{D}f_*\Psi(F)[d_X - d_Y]. \end{aligned}$$

For $\mathcal{L} \in \mathbf{D}_{\text{r-hol}}^b(\mathcal{D}_X)$, set

$$\Phi^0(\mathcal{L}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, \mathcal{O}_X).$$

Recall that $\Phi^0(\mathcal{L})$ is a \mathbb{C} -constructible complex of sheaves on X .

Proposition 4.2. *Let $\mathcal{L} \in \mathbf{D}_{\text{r-hol}}^b(\mathcal{D}_X)$, $\mathcal{M} \in \mathbf{D}_{\text{hol}}^b(\mathcal{D}_X)$ and $F \in \mathbf{D}_{\text{IR-c}}^b(\mathbb{IC}_{X \times \mathbb{P}})$. Then*

$$\begin{aligned} \Phi(\mathbb{D}(\mathcal{L} \otimes^{\mathbb{D}} \mathbb{D}\mathcal{M})) &\simeq R\mathcal{I}\mathcal{H}om(q^{-1}\Phi^0(\mathcal{L}), \Phi(\mathcal{M})), \\ \Psi(F \otimes q^{-1}\Phi^0(\mathcal{L})) &\simeq \Psi(F) \otimes^{\mathbb{D}} \mathcal{L}. \end{aligned}$$

Noticing that

$$\Phi(\mathcal{O}_X) \simeq \mathbb{C}_X \boxtimes R\mathcal{I}\mathcal{H}om(\mathbb{C}_{\{t \neq \infty\}}, \mathbb{C}_{\{\text{Re } t < ?\}}),$$

one checks easily that $\Psi(\Phi(\mathcal{O}_X)) \simeq \mathcal{O}_X$. Hence, Proposition 4.2 shows:

Theorem 4.3. (i) *For $\mathcal{L} \in \mathbf{D}_{\text{r-hol}}^b(\mathcal{D}_X)$, we have*

$$\begin{aligned} \Phi(\mathcal{L}) &\simeq q^{-1}\Phi^0(\mathcal{L}) \otimes \Phi(\mathcal{O}_X) \\ &\simeq \Phi^0(\mathcal{L}) \boxtimes R\mathcal{I}\mathcal{H}om(\mathbb{C}_{\{t \neq \infty\}}, \mathbb{C}_{\{\text{Re } t < ?\}}). \end{aligned}$$

(ii) *The morphism (3.1) is an isomorphism on $\mathbf{D}_{\text{r-hol}}^b(\mathcal{D}_X)$.*

(iii) *For any $\mathcal{L}, \mathcal{L}' \in \mathbf{D}_{\text{r-hol}}^b(\mathcal{D}_X)$, the natural morphism*

$$\text{Hom}_{\mathcal{D}_X}(\mathcal{L}, \mathcal{L}') \rightarrow \text{Hom}(\Phi(\mathcal{L}'), \Phi(\mathcal{L}))$$

is an isomorphism.

Therefore, Conjecture 3.1 holds true for regular holonomic \mathcal{D} -modules.

5. REVIEW ON GOOD FORMAL STRUCTURES

Let $D \subset X$ be a hypersurface. A flat meromorphic connection with poles at D is a holonomic \mathcal{D}_X -module \mathcal{M} such that $\text{ss}(\mathcal{M}) = D$ and $\mathcal{M} \simeq \mathcal{M}(*D)$.

We recall here the classical results on the formal structure of flat meromorphic connections on curves. (Analogous results in higher dimension have been obtained in [10, 9].)

Let X be an open disc in \mathbb{C} centered at 0.

For \mathcal{F} an \mathcal{O}_X -module, we set

$$\widehat{\mathcal{F}}|_0 = \widehat{\mathcal{O}}_{X,0} \otimes_{\mathcal{O}_{X,0}} \mathcal{F}_0,$$

where $\widehat{\mathcal{O}}_{X,0}$ is the completion of $\mathcal{O}_{X,0}$.

One says that a flat meromorphic connection \mathcal{M} with poles at 0 has a good formal structure if

$$(5.1) \quad \widehat{\mathcal{M}}|_0 \simeq \bigoplus_{i \in I} \left(\mathcal{L}_i \otimes^{\mathbb{D}} \mathcal{E}_{0|X}^{\varphi_i} \right) \widehat{|}_0$$

as $(\widehat{\mathcal{O}}_{X,0} \otimes_{\mathcal{O}_{X,0}} \mathcal{D}_{X,0})$ -modules, where I is a finite set, \mathcal{L}_i are regular holonomic \mathcal{D}_X -modules, and $\varphi_i \in \mathcal{O}_X(*0)$.

A ramification at 0 is a map $X \rightarrow X$ of the form $x \mapsto x^m$ for some $m \in \mathbb{N}$.

The Levelt-Turrittin theorem asserts:

Theorem 5.1. *Let \mathcal{M} be a meromorphic connection with poles at 0. Then there is a ramification $f: X \rightarrow X$ such that $\text{D}f^{-1}\mathcal{M}$ has a good formal structure at 0.*

Assume that \mathcal{M} satisfies (5.1). If \mathcal{M} is regular, then $\varphi_i \in \mathcal{O}_X$ for all $i \in I$, and (5.1) is induced by an isomorphism

$$\mathcal{M}_0 \simeq \bigoplus_{i \in I} \left(\mathcal{L}_i \otimes^{\mathbb{D}} \mathcal{E}_{0|X}^{\varphi_i} \right)_0.$$

However, such an isomorphism does not hold in general.

Consider the real oriented blow-up

$$(5.2) \quad \pi: B = \mathbb{R} \times S^1 \rightarrow X, \quad (\rho, \theta) \mapsto \rho e^{i\theta}.$$

Set $V = \{\rho > 0\}$ and let $Y = \{\rho \geq 0\}$ be its closure. If W is an open neighborhood of $(0, \theta) \in \partial Y$, then $\pi(W \cap V)$ contains a germ of open sector around the direction θ centered at 0.

Consider the commutative ring

$$\mathcal{A}_Y = R\mathcal{H}om_{\pi^{-1}\mathcal{D}_{\overline{X}}}(\pi^{-1}\mathcal{O}_{\overline{X}}, R\mathcal{H}om(\mathbb{C}_V, \mathcal{D}b_B^t)),$$

where \overline{X} is the complex conjugate of X .

To a \mathcal{D}_X -module \mathcal{M} , one associates the \mathcal{A}_Y -module

$$\pi^*\mathcal{M} = \mathcal{A}_Y \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M}.$$

The Hukuhara-Turrittin theorem states that (5.1) can be extended to germs of open sectors:

Theorem 5.2. *Let \mathcal{M} be a flat meromorphic connection with poles at 0. Assume that \mathcal{M} admits the good formal structure (5.1). Then for any $(0, \theta) \in \partial Y$ one has*

$$(5.3) \quad (\pi^* \mathcal{M})_{(0, \theta)} \simeq \left(\bigoplus_{i \in I} \pi^* (\mathcal{E}_{0|X}^{\varphi_i})^{m_i} \right)_{(0, \theta)},$$

where m_i is the rank of \mathcal{L}_i .

(Note that only the ranks of the \mathcal{L}_i 's appear here, since $x^\lambda (\log x)^m$ belongs to \mathcal{A}_Y for any $\lambda \in \mathbb{C}$ and $m \in \mathbb{Z}_{\geq 0}$.)

One should be careful that the above isomorphism depends on θ , giving rise to the Stokes phenomenon.

We will need the following result:

Lemma 5.3. *If \mathcal{M} is a flat meromorphic connection with poles at 0, then*

$$R\pi_*(\pi^* \mathcal{M}) \simeq \mathcal{M}.$$

6. RECONSTRUCTION THEOREM ON CURVES

Let X be a complex curve. Then Conjecture 3.1 (i) holds true:

Theorem 6.1. *For $\mathcal{M} \in \mathbf{D}_{\text{hol}}^b(\mathcal{D}_X)$ there is a functorial isomorphism*

$$(6.1) \quad \mathcal{M} \xrightarrow{\sim} \Psi(\Phi(\mathcal{M})).$$

Sketch of proof. Since the statement is local, we can assume that X is an open disc in \mathbb{C} centered at 0, and that $\text{ss}(\mathcal{M}) = \{0\}$.

By devissage, we can assume from the beginning that \mathcal{M} is a flat meromorphic connection with poles at 0.

Let $f: X \rightarrow X$ be a ramification as in Theorem 5.1, so that $\mathbf{D}f^{-1}\mathcal{M}$ admits a good formal structure at 0.

Note that $\mathbf{D}f_* \mathbf{D}f^{-1}\mathcal{M} \simeq \mathcal{M} \oplus \mathcal{N}$ for some \mathcal{N} . If (6.1) holds for $\mathbf{D}f^{-1}\mathcal{M}$, then it holds for $\mathcal{M} \oplus \mathcal{N}$ by Proposition 4.1, and hence it also holds for \mathcal{M} .

We can thus assume that \mathcal{M} admits a good formal structure at 0.

Consider the real oriented blow-up (5.2).

By Lemma 5.3, one has $\mathcal{M} \simeq R\pi_* \pi^* \mathcal{M}$. Hence Proposition 4.1 (or better, its analogue for π) implies that we can replace \mathcal{M} with $\pi^* \mathcal{M}$.

By Theorem 5.2, we finally reduce to prove

$$\mathcal{E}_{0|X}^\varphi \xrightarrow{\sim} \Psi(\Phi(\mathcal{E}_{0|X}^\varphi)).$$

Set $D' = \{x = 0\} \cup \{t = \infty\}$ and $U' = (X \times \mathbb{P}) \setminus D'$. By Proposition 2.1,

$$\mathbb{D}\mathcal{E}_{D'|X \times \mathbb{P}}^{t+\varphi} \simeq \mathbb{D}(\mathcal{E}_{0|X}^\varphi \boxtimes^{\mathbb{D}} \mathcal{E}_{\infty|\mathbb{P}}^t) \simeq \mathcal{E}_{D'|X \times \mathbb{P}}^{-t-\varphi}.$$

By Proposition 2.4, we thus have

$$\Phi(\mathcal{E}_{0|X}^\varphi) \simeq R\mathcal{I}\mathcal{H}om(\mathbb{C}_{U'}, \mathbb{C}_{\{\operatorname{Re}(t+\varphi) < ?\}}).$$

Noticing that $\Phi(\mathcal{E}_{0|X}^\varphi) \otimes \mathbb{C}_{D'} \in \mathbf{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_{X \times \mathbb{P}})$, one checks that $\Psi(\Phi(\mathcal{E}_{0|X}^\varphi) \otimes \mathbb{C}_{D'}) \simeq 0$.

Hence, Proposition 2.2 implies

$$\Psi(\Phi(\mathcal{E}_{0|X}^\varphi)) \simeq \Psi(\mathbb{C}_{\{\operatorname{Re}(t+\varphi) < ?\}}) \simeq \mathcal{E}_{0|\mathbb{C}}^\varphi.$$

□

Example 6.2. Let $X = \mathbb{C}$, $\varphi(x) = 1/x$ and $\mathcal{M} = \mathcal{E}_{0|X}^\varphi$. Then we have

$$H^k \Phi(\mathcal{M}) = \begin{cases} \mathbb{C}_{\{\operatorname{Re}(t+\varphi) < ?\}}, & \text{for } k = 0, \\ \mathbb{C}_{\{x=0, t \neq \infty\}} \oplus \mathbb{C}_{\{x \neq 0, t = \infty\}}, & \text{for } k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

7. STOKES PHENOMENON

We discuss here an example which shows how, in our setting, the Stokes phenomenon arises in a purely topological fashion.

Let X be an open disc in \mathbb{C} centered at 0. (We will shrink X if necessary.) Set $U = X \setminus \{0\}$.

Let \mathcal{M} be a flat meromorphic connection with poles at 0 such that

$$\widehat{\mathcal{M}}|_0 \simeq (\mathcal{E}_{0|X}^\varphi \oplus \mathcal{E}_{0|X}^\psi)|_0, \quad \varphi, \psi \in \mathcal{O}_X(*0).$$

Assume that $\psi - \varphi$ has an effective pole at 0.

The Stokes curves of $\mathcal{E}_{0|X}^\varphi \oplus \mathcal{E}_{0|X}^\psi$ are the real analytic arcs ℓ_i , $i \in I$, defined by

$$\{\operatorname{Re}(\psi - \varphi) = 0\} = \bigsqcup_{i \in I} \ell_i.$$

(Here we possibly shrink X to avoid crossings of the ℓ_i 's and to ensure that they admit the polar coordinate $\rho > 0$ as parameter.)

Since $\mathcal{E}_{0|X}^\varphi \simeq \mathcal{E}_{0|X}^{\varphi+\varphi_0}$ for $\varphi_0 \in \mathcal{O}_X$, the Stokes curves are not invariant by isomorphism.

The Stokes lines L_i , defined as the limit tangent half-lines to ℓ_i at 0, are invariant by isomorphism.

The Stokes matrices of \mathcal{M} describe how the isomorphism (5.3) changes when θ crosses a Stokes line.

Let us show how these data are topologically encoded in $\Phi(\mathcal{M})$.

Set $D' = \{x = 0\} \cup \{t = \infty\}$ and $U' = (X \times \mathbb{P}) \setminus D'$. Set

$$\begin{aligned} F_c &= \mathbb{C}_{\{\operatorname{Re}(t+\varphi) < c\}}, & G_c &= \mathbb{C}_{\{\operatorname{Re}(t+\psi) < c\}}, \\ F &= \mathbb{C}_{\{\operatorname{Re}(t+\varphi) < ?\}}, & G &= \mathbb{C}_{\{\operatorname{Re}(t+\psi) < ?\}}. \end{aligned}$$

By Proposition 2.4 and Theorem 5.2,

$$\Phi(\mathcal{M}) \simeq R\mathcal{I}\mathcal{H}om(\mathbb{C}_{U'}, H),$$

where H is an ind-sheaf such that

$$H \otimes \mathbb{C}_{q^{-1}S} \simeq (F \oplus G) \otimes \mathbb{C}_{q^{-1}S}$$

for any sufficiently small open sector S .

Let \mathfrak{b}^\pm be the vector space of upper/lower triangular matrices in $M_2(\mathbb{C})$, and let $\mathfrak{t} = \mathfrak{b}^+ \cap \mathfrak{b}^-$ be the vector space of diagonal matrices.

Lemma 7.1. *Let S be an open sector, and \mathfrak{v} a vector space, which satisfy one of the following conditions:*

- (i) $\mathfrak{v} = \mathfrak{b}^\pm$ and $S \subset \{\pm \operatorname{Re}(\psi - \varphi) > 0\}$,
- (ii) $\mathfrak{v} = \mathfrak{t}$, $S \supset L_i$ for some $i \in I$ and $S \cap L_j = \emptyset$ for $i \neq j$.

Then, for $c' \gg c$, one has

$$\operatorname{Hom}((F_c \oplus G_c)|_{q^{-1}S}, (F_{c'} \oplus G_{c'})|_{q^{-1}S}) \simeq \mathfrak{v}.$$

In particular,

$$\operatorname{End}((F \oplus G) \otimes \mathbb{C}_{q^{-1}S}) \simeq \mathfrak{v}.$$

This proves that the Stokes lines are encoded in H . Let us show how to recover the Stokes matrices of \mathcal{M} as glueing data for H .

Let S_i be an open sector which contains L_i and is disjoint from L_j for $i \neq j$. We choose S_i so that $\bigcup_{i \in I} S_i = U$.

Then for each $i \in I$, there is an isomorphism

$$\alpha_i: H \otimes \mathbb{C}_{q^{-1}S_i} \simeq (F \oplus G) \otimes \mathbb{C}_{q^{-1}S_i}.$$

Take a cyclic ordering of I such that the Stokes lines get ordered counterclockwise.

Since $\{S_i\}_{i \in I}$ is an open cover of U , the ind-sheaf H is reconstructed from $F \oplus G$ via the glueing data given by the Stokes matrices

$$A_i = \alpha_{i+1}^{-1} \alpha_i|_{q^{-1}(S_i \cap S_{i+1})} \in \mathfrak{b}^\pm.$$

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